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STANDARD MODEL LAGRANGIAN

2.1 INTRODUCTION

The standard model of Elementary Particle Physics describes with amazing parsimony (only 19 parameters!) all known interactions over the scales that have been explored by experiments: from the Hubble radius of $10^{30}$ cm all the way down to scales of the order of $10^{-16}$ cm. Together with cosmological initial conditions when the universe was much tinier, the standard model is believed to encode the information necessary to deduce (in principle) all observed physical phenomena, life forms, etc...; its remarkable internal consistency is both a source of wonder and despair to those who seek to predict the future based on its flaws. Its structure does not run counter to any of our (rather incomplete) theoretical prejudices, and as a renormalizable model, it does not manifestly require new physics at a higher energy scale.

There are few obvious chinks in the armor of the standard model. Some are theoretical, suggested by the behavior of the standard model’s parameters in the ultraviolet. Depending on the value of the mass of the Higgs particle, one can lose perturbative control in the Higgs self coupling close to experimental scales, suggesting the appearance of new interactions and/or new particles at those energies. The standard model also has Landau poles in its gauge and Yukawa couplings, but they typically occur at length scales much shorter than that at which quantum corrections to Einstein’s theory of gravity are expected. In the absence of a satisfactory quantum theory of gravity, it is difficult to assess the meaning of these poles. On the other hand, the unification of the standard model to include gravity is the central conceptual problem of our time. Fortunately it need not concern us for the purposes of this book, as the scales of interest are at least seventeen orders of magnitudes smaller than the smallest scale hitherto explored! Only one candidate theory, superstring theory, addresses this question with any de-
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gree of success, but it only presents a qualitative picture which cannot yet be tested by experiment. The situation is reminiscent of the problem of the self-stress of the electron in classical electrodynamics: it tells you something is wrong with the theory, but it does not bring about an immediate confrontation with experiment.

The origin of different scales in nature is not answered by the standard model. There is but one fundamental constant with the dimension of length: Gravity comes with its own scale, the Planck length $\ell_{\text{Pl}}$. All units of length should be scaled to it. In human terms, it corresponds to a very small distance indeed, $10^{-33}$ cm. In supernatural (Planck) units, the Planckton?, the length scales of the standard model are enormous:

$$\frac{\ell_N}{\ell_{\text{Pl}}} \sim \frac{10^{-13}}{10^{-33}} \sim 10^{20} \quad \text{and} \quad \frac{\ell_F}{\ell_{\text{Pl}}} \sim \frac{10^{-16}}{10^{-33}} \sim 10^{17},$$

where $\ell_N, \ell_F$ are the sizes associated with nuclear and weak interactions, respectively. It is a source of great intellectual worry that the standard model appears to be consistent at a scale that is so very different from the Planck scale. Naive expectations are for all phenomena to occur at their natural scale which, in fundamental physics, is the Planck scale. The understanding of these large numbers is probably the most fundamental question in microphysics today.

Today, there are no contradictions between particle physics experiments at accelerators and the standard model. Not only is it standing up to experiments in its gross features but also remarkably in numerous consistency checks that test its very quantum nature. A recent singular exception comes from non-accelerator experiments, which offers evidence for neutrino masses through flavor oscillations of neutrinos produced by cosmic rays. Further evidence from the measurements of neutrino fluxes from the Sun also suggest neutrinos masses through oscillations. This conflicts with the standard model in which the neutrinos which accompany the charged leptons in $\beta$ decay are strictly massless. If these results are reproduced by newer neutrino detectors soon to be deployed, they provide the first true signal of physics beyond the standard model.

There may also be a possible conflict with cosmology. The standard model does contain the capability, through its anomaly, of generating a predominance of baryons over antibaryons in the universe. However, in its present formulation, it does not seem capable of reproducing the required amount of baryon asymmetry. This may be only a quantitative failure, tied to our
incomplete understanding of the kinetics of the electroweak phase transition in the early universe.

Another arena of possible confrontation is the nature of dark matter. Observations indicate the presence of a dominant amount of non-luminous matter in the universe, so much so in fact that we should think of luminous matter as the foam that rides the crest of waves of dark matter in the cosmic ocean. The standard model, which contains no exotic matter, predicts all clustering neutral matter, dark and luminous, to be baryonic, while nucleosynthesis puts an upper bound on the amount of baryonic matter. As this bound seems to be exceeded by observation, the standard model must be extended to incorporate dark matter. This discussion does not include the prevailing theoretical prejudices based on inflation which predicts that the universe is at critical density, well above the nucleosynthesis bound. In this picture dark matter must be made up of type(s) of matter not in the standard model. While it is fair to say that the evidence for generalizing the standard model is most compelling in this domain, our knowledge of both the cosmological scenario before nucleosynthesis and of the late matter abundances in the universe are too nebulous to convincingly demonstrate the necessity of physics beyond the standard model.

Finally, in spite of its great success in confronting experiments, the standard model is not aesthetically satisfying: it contains particles of different spin, three disconnected gauge symmetries, two unexplained scales (nuclear size and neutron lifetime), as well as seventeen additional parameters, ranging over nine orders of magnitude.

Therefore it should not be too surprising if the standard model is not widely viewed as fundamental, but rather as the broken pieces of a more integrated structure. The hope of physicists is to glue its parts together into one theoretical construct with few if any undetermined parameters. Models for such a hypothetical structure abound in the theoretical literature, but they typically introduce many parameters, often more in fact than those they seek to explain! Although some of these models have definite theoretical appeal, their acceptance must await the verdict of experiments. While any proposal which reproduces even a modicum of already known facts can become an instant candidate for the Theory Of Everything (TOE), we do not yet seem to have enough information to reconstruct the puzzle: the standard model is not sufficient to put Humpty-Dumpty back together again. It is tempting to think that such a unified structure may have emerged unscathed at the earliest moments after the Big Bang, only to be broken in the course of cosmological evolution into the chiral shards we observe today.
Thus physicists who seek the TOE may be regarded as archeologists who
probe our distant past when we were all in the same cosmic soup!

Since it is doubtful that direct experimental evidence can be obtained
about physics at the Planck length, we have to restrain our theoretical am-
bitions. In this book, we consider extensions of the standard model which
affect scales that are attainable by experiments in the near future. Hence, we
present first the standard model, and then some of its minimal low-energy
extensions, hoping to gather enough new shards to enable us to reconstruct
the mother structure by the end of this book.

This chapter contains, after a short historical sketch, a self-contained sum-
mary of the standard model of Weak, Electromagnetic and Strong Interac-
tions.

2.2 Historical Preamble

Our presentation of the standard model does not follow historical lines, since
its starting point is the standard model Lagrangian. Yet, its final form is
the result of the inspired work of many experimentalists and theorists, over
a period of seventy years or so. It is not possible to do justice to their
contributions in the short description that follows. For detailed accounts,
the reader is referred to various books and collection of reprints which are
listed at the end of this book. Suffice it to say that the history of the
standard model is remarkably rich as it mirrors the scientific effervescence
of the natural sciences in this century.

The first third of the XXth century witnessed unparalleled scientific ac-
tivities, spurred on by the dramatic experimental discoveries which resulted
in the establishment of quantum mechanics, and general relativity. These
set the stage for the intense bursts of experimental and theoretical break-
throughs, that led five decades later to the formulation of the standard model
of the fundamental interactions. Below we present a decade by decade es-
quisse of these amazing developments.

• 1930's In this decade many building blocks of the standard model were
identified: its interactions, and its first family of elementary particles. At
its onset, P. A. M. Dirac formulated one basic ingredient of the standard
model, Quantum Electrodynamics, which describes the interaction of pho-
tons with matter. In 1933, E. Fermi generalized this work to include β decay,
incorporating the neutrino, that had been postulated a few years earlier by
W. Pauli, and the neutron, recently discovered by J. Chadwick. In 1935, H.
Yukawa understood the short range of the strong interactions in terms of the
exchange of a massive (elementary) particle. In 1937, a massive elementary particle was discovered in cosmic rays, but it interacted too weakly to be Yukawa’s particle; it was the muon, the first member of the second family of elementary particles. The second half of the decade was spent confronting the theoretical difficulties associated with QED, but ended with O. Klein’s remarkable anticipation of gauge theories as the root cause of $\beta$ decay.

- **1940’s** The computational rules of QED are understood by Feynman, Schwinger and Tomonaga, in terms of renormalization. Calculations and measurements of the anomalous magnetic moment of the electron and the Lamb shift provide spectacular evidence for their approach. A similar simplicity evades all formulations of the strong and weak interactions, although experiments begin to show impressive structures. Yukawa’s true particle, the $\pi$ meson is discovered in 1947, and soon after the first of a class of new particles, which we now call the $K$-mesons.

- **1950’s** This is probably a spectacular decade in the history of the standard model. Its beginning saw the discovery of the $\Lambda$ particle. These new strongly-interacting particles are always produced in pairs, with slow decay rates, features interpreted by M. Gell-Mann as evidence of strangeeness, a new quantum number. The second half of the decade was just as impressive: neutrinos were detected in 1956 by C. Cowan and F. Reines. T. D. Lee and C. N. Yang proposed experiments to test parity violation by the weak interactions. Experimental verifications by C. S. Wu and by V. Telegdi soon followed. Soon thereafter, R. Marshak and E. C. G. Sudarshan and then R. Feynman and M. Gell-Mann identified the source of parity violation in terms of the $V-A$ vector and axial vector interactions. Spurred on by the 1954 work of C. N. Yang and R. L. Mills, theorists under the influence of J. Schwinger and others, soon realized the possibility that weak interactions could be caused by a massive vector particle, the W-boson.

- **1960’s** This was the decade of great theoretical synthesis. First, the zoo of strongly interacting particles was organized in terms of $SU(3)$, by Y. Ne’eman and M. Gell-Mann at the beginning of the 1960’s. A few years later, M. Gell-Mann and, independently G. Zweig, used this classification to postulate quarks as the building blocks of baryonic matter. The addition of spin to this synthesis by F. Gürsey and L. Radicati led to a new puzzle, interpreted by O. W. Greenberg in terms of parastatistics for hadrons. This led Y. Nambu in 1964 to propose that quarks interact with one another through a $SU(3)$ Yang-Mills theory. The beginning of this decade saw the emergence of the electroweak formulation of the standard model in its mod-
ern form, starting with the work of S. Glashow in 1961, then A. Salam and J.C. Ward, in 1964, and finally S. Weinberg in 1967. These works resulted in a model of the electroweak interactions of leptons, which after spontaneous breaking of the electroweak symmetry, reduces to QED. They predicted a new force, mediated by a neutral vector particle, the $Z$ boson, and used ideas that originated in the BCS theory of superconductivity. Applied by Nambu to pion physics in 1961, and generalized by J. Goldstone, spontaneous breaking was shown by P. W. Higgs and R. Brout and F. Englert to generate massive gauge bosons from massless ones when the symmetry is gauged.

There were experimental surprises as well. The muon neutrino is discovered in 1962 by G. Danby et al in 1964, J. Cronin and V. Fitch discover CP violation. At the end of the decade a surprising scaling behavior (anticipated by J. Bjorken in 1966) is found in deep inelastic scattering experiments, implying the existence of hard constituents inside protons.

- **1970’s** In 1970, S. Glashow, J. Iliopoulos and L. Maiani added theoretical evidence for a second family, by showing that a fourth *charm* quark could explain the absence of flavor-changing neutral interactions. This is followed by the theoretical breakthrough of G. ’t Hooft, a student under the *aegis* of M. Veltman, who showed that the Glashow-Salam-Ward-Weinberg model was in fact renormalizable. C. Bouchiat, J. Iliopoulos and P. Meyer showed how quantum consistency required the existence of both quarks and leptons to cancel the Adler-Bell-Jackiw *anomaly* constraints. M. Gell-Mann and H. Fritzsch, interpreted the parastatistics as evidence for a new quantum number for quarks, *color*, and related it to the value of the $\pi^0$ lifetime. The modern version of quark dynamics, Quantum Chromodynamics, was finally formulated by M. Gell-Mann, H. Fritzsch and H. Leutwyler in 1973. G. ’t Hooft, and then H. Politzer, and D. Gross and F. Wilczek showed that QCD was asymptotically-free, explaining the remarkable scaling results found by experiments in the late sixties. This showed that all interactions, strong, weak and electromagnetic were due to Yang-Mills vector exchange theories. In 1973, M. Kobayashi and T. Maskawa point out that a third family naturally implies CP-violation. In 1974, the $J/\Psi$ charmed quark-antiquark bound state is discovered at Brookhaven and SLAC, completing the first two families of elementary particles. But there is more: in 1975, M. Perl discovers the lepton member of the third family, the $\tau$ lepton. The bottom quark is soon thereafter discovered at FermiLab. Neutral current interactions are discovered in 1976 at CERN, and in 1978, the parity structure of the neutral current is determined to be that predicted in the 1960’s. By 1979, the final
2.3 The Lagrangian

The Lagrangian of the standard model is made up of four different parts, each named after illustrious scientists:

\[ \mathcal{L}_{SM} = \mathcal{L}_{YM} + \mathcal{L}_{WD} + \mathcal{L}_{Yu} + \mathcal{L}_{H}. \] (2.1)

The first is the Yang-Mills part \( \mathcal{L}_{YM} \), which describe the low energy gauge groups of the standard model, \( SU_3 \) for color, \( SU_2 \) for weak isospin, and \( U_1 \) for hypercharge.

\[ \mathcal{L}_{YM} = \mathcal{L}_{QCD} + \mathcal{L}_{Iw} + \mathcal{L}_Y, \]
\[ = -\frac{1}{4g_3^2} \sum_{A=1}^{8} G_{A \mu \nu} G^{A \mu \nu} - \frac{1}{4g_2^2} \sum_{a=1}^{3} F_{a \mu \nu} F^{a \mu \nu} - \frac{1}{4g_1^2} B_{\mu \nu} B^{\mu \nu}. \] (2.2)
where \( g_1, g_2, g_3 \) are the dimensionless coupling constants, corresponding to color, weak isospin and hypercharge, respectively. The color field strengths are given by

\[
G^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - f^{ABC} A^B_\mu A^C_\nu, \quad A, B, C = 1, ..., 8, \quad (2.3)
\]

with \( A^B_\mu \) represent the eight gluon fields, and \( f^{ABC} \) are the structure functions of \( SU_3 \). The weak isospin field strengths

\[
F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu - \epsilon^{abc} W^b_\mu W^c_\nu, \quad a, b, c = 1, 2, 3, \quad (2.4)
\]

are written in terms of the intermediate vector bosons \( W^a_\mu \) and the \( SU_2 \) structure function \( \epsilon^{abc} \). Finally, the Abelian hypercharge field strength is

\[
B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \quad (2.5)
\]

The second part of the Lagrangian, \( L_{WD} \), describes the fermion fields and their gauge interactions. The fermions in the standard model can be split into two categories, quarks which are triplets under the color gauge groups and leptons which have no color. Within each category some transform as weak doublets, some as weak singlets. Specifically, we have (all fermions are represented by two-component Weyl left-handed fields)

- weak doublet of leptons: \( L_i = \begin{pmatrix} \nu_i \\ \ell_i \end{pmatrix} \sim (2, 1^c)^{y_1} \)
- lepton weak singlets: \( \bar{\ell}_{iL} \sim (1, 1^c)^{y_2} \)
- quark weak doublet: \( Q_i = \begin{pmatrix} u_i \\ d_i \end{pmatrix} \sim (2, 3^c)^{y_3} \)
- antiquark weak singlets: \( \bar{u}_{iL} \sim (1, \bar{3}^c)^{y_4} \)
- antiquark weak singlets: \( \bar{d}_{iL} \sim (1, \bar{3}^c)^{y_5} \)

The notation denotes the color, isospin, and hypercharge assignments, in the form \( (SU_2^W, SU_3^c)_Y \). The index \( i \), ranges over the three families of chiral fermions. In terms of the matrices

\[
W^a_\mu = \frac{1}{2} W^a_\mu (x) \tau^a, \quad A^A_\mu = \frac{1}{2} A^A_\mu (x) \lambda^A, \quad (2.5)
\]

where \( \tau^a \) and \( \lambda^A \) are the Pauli and Gell-Mann matrices for \( SU_2^W \) and \( SU_3^c \), we can easily express the covariant derivatives on the various fermion fields.
\[ D_\mu L_i = (\partial_\mu + iW_\mu + \frac{i}{2}y_1B_\mu)L_i , \]
\[ D_\mu \bar{e}_i = (\partial_\mu + \frac{i}{2}y_2B_\mu)\bar{e}_i , \]
\[ D_\mu Q_i = (\partial_\mu + iA_\mu + iW_\mu + \frac{i}{2}y_3B_\mu)Q_i , \]
\[ D_\mu \bar{u}_i = (\partial_\mu - iA_\mu^* + \frac{i}{2}y_4B_\mu)\bar{u}_i , \]
\[ D_\mu \bar{d}_i = (\partial_\mu - iA_\mu^* + \frac{i}{2}y_5B_\mu)\bar{d}_i . \]

The factor \( \frac{1}{2} \) in front of the hypercharge field is conventional. This whole gauge structure is family independent. In terms of the above, the Weyl-Dirac Lagrangian is given by

\[ L_{WD} = \sum_i \left( L_i^\dagger \sigma^\mu D_\mu L_i + \bar{e}_i^\dagger \sigma^\mu D_\mu \bar{e}_i + Q_i^\dagger \sigma^\mu D_\mu Q_i + \bar{u}_i^\dagger \sigma^\mu D_\mu \bar{u}_i + \bar{d}_i^\dagger \sigma^\mu D_\mu \bar{d}_i \right) . \] (2.6)

The partial Lagrangian \( L_{YM} + L_{WD} \) displays large global symmetries. To start, there are several different family chiral symmetries, one for each set of fermions with the same quantum numbers. For example, the global transformation on the lepton doublet

\[ L_i \rightarrow L'_i = U_{ij}L_j , \] (2.7)

where \( U_{ij} \) is a \( 3 \times 3 \) unitary matrix, which leaves the partial Lagrangian invariant. Since this invariance is true for all the different types of fermions we have the global family symmetry \( U(3) \times U(3) \times U(3) \times U(3) \times U(3)! \) It is actually a wee bit smaller since two of the chiral \( U_1 \) currents are anomalous. We will address this question later on.

Much of this enormous global symmetry is explicitly broken by the Yukawa interactions of the standard model, by which fermion pairs interact with spinless particles. Let us therefore examine the quantum numbers of the two fermion operators and look for recurrences. The Lorentz scalar, color singlet, weak doublets occur in the combinations

\[ \Delta I_w = \frac{1}{2} : \hat{Q}u , \hat{Q}d , \hat{L}e , \] (2.8)
where we have not shown the family indices, and have used the notation introduced in the previous chapter $\hat{\eta} \equiv \eta^T \sigma_2$.

Classically, the hypercharge assignments are arbitrary, but at the quantum level they are constrained by the Adler-Bell-Jackiw anomaly, which spoils the renormalizability of the theory whenever it spoils the conservation of a gauged current. The Adler-Bardeen theorem asserts that the anomaly occurs only at $\mathcal{O}(\hbar)$ (one-loop order). It is therefore sufficient to demand that the relevant one loop triangle graphs vanish. The color group $SU_3^c$ does not have any chiral anomaly, since there are as many (left-handed) quarks as antiquarks – it is vector-like. The weak isospin group $SU_2^W$ is anomaly-free because of its structure: it has no $d$-coefficient. This leaves the hypercharge $U_1^Y$ as the only possible candidate for gauged anomalies. These would appear in the following three triangle graphs (the internal lines represent left-handed fermions)

These vanish if the following equations are satisfied,

\[ 2y_3 + y_4 + y_5 = 0 \quad y_1 + 3y_3 = 0 \]  

\[ 2y_1^3 + y_2^3 + 3(2y_3^3 + y_4^3 + y_5^3) = 0 \]

They relate both quark and lepton hypercharges, providing the first hint of some sort of unification: two hypercharges are still undetermined, but one can be normalized arbitrarily by redefining the hypercharge coupling constant $g_1$. The vanishing of the mixed $SU(3) - U(1)$ anomaly condition requires the two quark bilinears to have opposite value of hypercharge.
We may consider another type of anomaly: the mixed gravitational anomaly which is generated by a graph of the form

![Graph](image)

with one hypercharge gauge boson and two gravitons. Its vanishing results in the additional condition

$$2y_1 + y_2 + 6y_3 + 3y_4 + 3y_5 = 0 \quad (2.9)$$

This anomaly condition is not on the same footing as the others since it brings in quantum gravity, which is not a renormalizable theory: it would affect the conservation of the hypercharge current only if graviton loops are taken seriously. Nevertheless it is likely to be present in a renormalizable theory of gravity. In the absence of Yukawa couplings, the mixed gravitational anomaly, together with the ABJ anomalies, yield the relation

$$y_2 = 6y_3 \quad (2.10)$$

If we substitute this equation in the unmixed $U(1)$ anomaly condition, we obtain

$$18y_3(2y_3 - y_5)(4y_3 + y_5) = 0 \quad (2.11)$$

It seems that we have three physically different solutions, but this is illusory: the second solution $y_5 = 2y_3$, and the third solution $y_5 = -4y_3$ can be seen to be the same with $y_4$ and $y_5$ interchanged. This is not surprising since these two fields are only distinguished by their hypercharges. We really have only two solutions, $y_3 = 0$ and $y_5 = 2y_3$. Of these, the simplest $y_3 = 0$, is quite uninteresting: none of the three two fermion combinations have the same hypercharge. For the other solution, $y_5 = 2y_3$, $\bar{L}\bar{e}$ and $\bar{Q}\bar{d}$ have the same hypercharge. Which one does nature choose? It is the Yukawa couplings that settle the issue.

It is remarkable that one of these solutions is realized by the Yukawa couplings. Under the $SU_2^W \times U_1^Y$ gauge group, potential color singlet quark
masses violate weak isospin by half units, \( \Delta I_w = \frac{1}{2} \). In order to generate masses for the charged fermions, without violating the renormalizability of the theory, one introduces a spinless boson Higgs field \( H \), transforming as a weak doublet, color singlet, and with hypercharge \( y_h \):

\[
H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \sim (1^c, 2)_{y_h}.
\]

With only this field, all the gross features of the data can be explained. The following Yukawa couplings can be added

\[
\mathcal{L}_{Y_\mu} = i \tilde{L}_i \bar{e}_j H^* Y_{ij}^{[e]} + i \tilde{Q}_i \bar{d}_j H^* Y_{ij}^{[d]} + i \tilde{Q}_i \bar{u}_j \tau_2 H Y_{ij}^{[u]} + \text{c.c.} ,
\]

where \( \tau_2 \) are the weak isospin indices and the Yukawa coupling constants are unknown complex \( 3 \times 3 \) matrices \( Y^{[e]} \), \( Y^{[d]} \), and \( Y^{[u]} \). Invariance under hypercharge transformations implies the further relations

\[
y_h = y_1 + y_2 = -(y_3 + y_4) = (y_3 + y_5) .
\]

These Yukawa couplings determine all the hypercharge assignments to be (in units of \( y_h \)),

\[
y_1 = -1, \ y_2 = +2, \ y_3 = +\frac{1}{3}, \ y_4 = -\frac{4}{3}, \ y_5 = +\frac{2}{3} .
\]

Hence, the absence of anomalies from the standard model requires hypercharge to be quantized. It is noteworthy that although the standard model does not include gravitational interactions, the mixed gravitational anomaly of the hypercharge happens to vanish.

The global symmetries of \( \mathcal{L}_{YM} + \mathcal{L}_{WD} \), can be used to simplify the Yukawa couplings. We can write without loss of generality any matrix as the product of a unitary matrix times a real diagonal matrix times another unitary matrix, yielding for the lepton Yukawa matrix (not showing the family indices)

\[
Y^{[e]} = U_e^t M^{[e]} V_e ,
\]

where \( U_e \) and \( V_e \) are \( 3 \times 3 \) unitary matrices,

\[
U_e U_e^\dagger = V_e V_e^\dagger = 1 ,
\]

and \( M^{[e]} \) is a real diagonal matrix. We can absorb the unitary matrices \( U_e \) and \( V_e \) through the field redefinitions
2.3 The Lagrangian

\[ L' = U_e L , \quad \bar{e}' = V_e \bar{e} , \]  

(2.16)

without affecting the rest of the Lagrangian \( \mathcal{L}_Y M + \mathcal{L}_{WD} \). Dropping the primes, the leptonic part of the Yukawa couplings becomes flavor-diagonal

\[ i \sum_{i=1}^{3} \bar{L}_i \bar{e}_i H^* y_{ii}^{[e]} + \text{c.c.} , \]

where \( y_{ii}^{[e]} \) are the diagonal elements of \( M^{[e]} \). The lepton Yukawa couplings therefore break the leptonic global symmetry \( SU_3 \times SU_3 \times U_1 \times U_1 \) down to three phase transformations,

\[ L_i \rightarrow e^{i\alpha_i} L_i , \quad \bar{e}_i \rightarrow e^{-i\alpha_i} \bar{e}_i , \]  

(2.17)

interpreted as the three lepton numbers, one for each family (electron-, muon-, and tau-numbers).

We can attempt similar simplifications in the quark Yukawa sector by setting

\[ Y^{[d]} = U_d^t M^{[d]} V_d , \]  

(2.18)

\[ Y^{[u]} = U_u^t M^{[u]} V_u , \]  

(2.19)

where \( U_{u,d} \) and \( V_{u,d} \) are unitary family matrices; \( M^{[d]} \) and \( M^{[u]} \) are diagonal and real.

We note immediately that the unitary matrices \( V_u \) and \( V_d \) can be absorbed in a redefinition of the fields

\[ \bar{u} \rightarrow V_u \bar{u} , \quad \bar{d} \rightarrow V_d \bar{d} . \]  

(2.20)

On the other hand, the matrices \( U_u \) and \( U_d \) cannot both be absorbed away by field redefinitions. Indeed, the disappearance of \( U_u \) in one Yukawa coupling forces its reappearance as \( U_u^\dagger \) in the other one. Similarly if \( U_d \) disappears from the second, it reappears as \( U_d^\dagger \) in the first. The best we can do is to rewrite the quark Yukawa couplings in the simplified form

\[ i \hat{Q}_i \bar{d}_i H^* y_{ii}^{[d]} + i \hat{Q}_i (\mathcal{V})_{ji} \bar{u}_j \tau_2 H y_{jj}^{[u]} , \]

where

\[ \mathcal{V} = U_u U_d^\dagger , \]  

(2.21)
is a unitary matrix. One can make a final modification by expurgating yet more redundant phases from $\mathcal{V}$. Indeed, any unitary matrix can be decomposed as

$$\mathcal{V} = \mathcal{P}^T U \mathcal{P}',$$  \hspace{1cm} (2.22)

where $\mathcal{P}$ and $\mathcal{P}'$ are diagonal phase matrices generated by elements of the Cartan subalgebra, and $U$ contains the remaining parameters. For a rotation matrix, this is exactly the Euler decomposition. In the physical case of three families, $\mathcal{V}$ depends on nine parameters, one of which is an overall phase. The other eight correspond to the generators of $SU_3$. Using this decomposition (called Iwasawa in mathematics), we see that $\mathcal{P}$ and $\mathcal{P}'$ each depend on two parameters, leaving the unitary $U$ with four parameters. Three correspond to real rotations while the fourth one must be a phase, not in the $SO(3)$ subgroup of $SU_3$. Why is this decomposition useful? We see that $\mathcal{P}$ can be absorbed in $\bar{u}$, and $\mathcal{P}'$ can be absorbed in $Q$ and then, through the first coupling, transferred to $\bar{d}$, where it finally disappears, leaving us with the final form of the Yukawa couplings of the standard model

$$\mathcal{L}_{Yu} = i \left( \hat{L}_i y_{ei}^i \bar{e}_i + \hat{Q}_i y_{hid}^i \bar{d}_i \right) H^* + i \hat{Q}_i U_{ji} y_{dji}^i \bar{u}_j \tau_2 H + \text{c.c.} \hspace{1cm} (2.23)$$

Let us remark that if there had been only two families, the Euler decomposition would have left us with only one rotation angle, and no phase. With three families, the Yukawa sector of the standard model depend on thirteen (real) parameters: nine masses, three mixing angles and one phase. The matrix $U$ is called the Cabibbo-Kobayashi-Maskawa (CKM) matrix. In the quark sector, the only remnant of the global symmetries of $\mathcal{L}_{WD}$ is a common phase transformation for all the quarks,

$$Q_i \rightarrow e^{i\alpha} Q_i \, , \, \bar{u}_i \rightarrow e^{-i\alpha} \bar{u}_i \, , \, \bar{d}_i \rightarrow e^{i\alpha} \bar{d}_i \hspace{1cm} (2.24)$$

and it corresponds to quark number ($1/3$ of baryon number). To summarize, the Yukawa couplings have reduced the global symmetries of the gauged kinetic terms to four phase symmetries, baryon number and the three lepton numbers. However it is easy to see (see problem) that of these one is anomalous, leaving only three truly conserved quantum numbers.

We now turn to the fourth and final part of the standard model Lagrangian which describes the Higgs doublet, its interactions with the gauge fields, and with itself. The gauge-covariant derivative acting on the Higgs takes the form
2.3 The Lagrangian

\begin{equation}
\mathcal{D}_\mu H = (\partial_\mu + iW_\mu + \frac{i}{2} y_B B_\mu)H.
\end{equation}

The Higgs Lagrangian is simply given by

\begin{equation}
\mathcal{L}_H = (\mathcal{D}_\mu H)^\dagger (\mathcal{D}^\mu H) - V(H),
\end{equation}

where \( V \) is the most general renormalizable potential invariant under \( SU_2^W \times U_1^Y \); it is given by

\begin{equation}
V = -\mu^2 H^\dagger H + \lambda (H^\dagger H)^2.
\end{equation}

It can be shown that there is only one quartic polynomial with this invariance. The dimensionless coupling \( \lambda \) is taken to be positive to insure that \( V \) is bounded from below, and \( \mu^2 \) is the only parameter with units of mass in the classical Lagrangian. The sign in front of the mass term is taken to be negative for reasons that will soon become obvious.

The alert reader has already noticed that \( V(H) \) is invariant under a much larger symmetry, \( SO_4 \), acting on the four real components of the complex doublet. This symmetry is broken by other parts of the Lagrangian. Such a symmetry, shared only by part of the Lagrangian is sometimes called accidental. We have already encountered such symmetries in the Weyl-Dirac part of the Lagrangian. Since \( SO_4 \sim SU_2^L \times SU_2^R \), the four vector made up of the four real components of the Higgs transforms as \((2,2)\). The first \( SU_2^L \) is the gauged weak isospin. The diagonal sum of these two \( SU(2)s \) is sometimes called “custodial”; it is broken both by the Yukawa couplings, and by the couplings to hypercharge.

We have now come to the end of our description of the Lagrangian of the standard model. We have seen that it depends on the three coupling constants \( g_1, g_2, g_3 \) of the gauge groups, on the 13 parameters of the Yukawa couplings (12 real + 1 phase), on the Higgs self coupling \( \lambda \), and on one dimensionful “coupling” \( \mu \). Thus, the classical standard model depends on 18 parameters. In addition to being invariant under the gauge group \( SU(3) \times SU(2) \times U(1) \), the standard model is (at the classical level) invariant under four global phase transformations, the three lepton numbers and baryon number.

We have stated that this counting is altered by quantum effects. Quantum anomalies break a linear combination of baryon and lepton numbers. As we shall investigate in detail, this also introduce another parameter that
describes the QCD vacuum. These anomalies, however, do not alter the renormalizability of the theory since they affect only ungauged symmetries.

We conclude this description of the standard model Lagrangian by discussing its discrete symmetries, in particular its CP properties. Of course, the locality and reality of the Lagrangian implies the invariance of the standard model under the combined operation of CPT. The chiral character of weak isospin obviously breaks parity, but the CP analysis is more subtle.

We start with the transformation of a generic fermion field under CP:

$$ CP : \psi_L \rightarrow \sigma_2 \psi_L^\dagger . $$

(2.28)

This transformation leaves the kinetic term invariant. Suppose $\psi_L$ forms a representation of some gauge group. The fermion gauge interaction is

$$ L_{\text{int}} = ig \psi_L^\dagger \sigma_\mu \mathbf{A}_\mu \psi_L , $$

(2.29)

where $\mathbf{T}$ are the matrices which represent the group in the representation of $\psi_L$. Under CP, the currents transform as

$$ i\psi_L^\dagger \mathbf{T}_L \psi_L \rightarrow i\psi_L^T \mathbf{T}_L \sigma_2 \psi_L^* = -i\psi_L^\dagger \mathbf{T}_L \psi_L , $$

(2.30)

$$ i\psi_L^\dagger \mathbf{\sigma_T} \psi_L \rightarrow i\psi_L^T \sigma_2 \mathbf{\sigma_T} \psi_L^* = +i\psi_L^\dagger \mathbf{\sigma_T} \psi_L . $$

(2.31)

Thus invariance of this coupling is assured if we demand

$$ A_S^0 \rightarrow -A_S^0 , \quad A_S^i \rightarrow +A_S^i , $$

(2.32)

$$ A_A^0 \rightarrow +A_A^0 , \quad A_A^i \rightarrow -A_A^i , $$

(2.33)

where the superscripts S(A) denote the generators for which the $\mathbf{T}$ matrices are symmetric or antisymmetric. In $SU_2$ for example, the set $S$ contains $\tau^3$ and $\tau^1$ while the $A$ set contains $\tau^2$. This is consistent with the naive expectation that charge conjugation changes the sign of $i$ since the $\mathbf{T}$-matrices are hermitian. Also we recover the desired transformations under CP since the space components are odd under Parity. Further, under CP, the field strengths transform as

$$ F_{0i}^S \rightarrow +F_{0i}^S , \quad F_{ij}^S \rightarrow -F_{ij}^S , $$

(2.34)
2.3 The Lagrangian

\[ F_{0i}^A \rightarrow -F_{0i}^A, \quad F_{ij}^A \rightarrow +F_{ij}^A, \]  
which leaves the Yang-Mills Lagrangian, \( \vec{F}_{\mu \nu} \cdot \vec{F}^{\mu \nu} \), invariant. Note for future reference that the surface term

\[ \epsilon^{\mu \nu \rho \sigma} \vec{F}_{\mu \nu} \cdot \vec{F}_{\rho \sigma} \sim \epsilon^{ijk} \vec{F}_{0i} \cdot \vec{F}_{jk}, \]  
is clearly odd under CP and P.

A generic Yukawa contribution to the Lagrangian, involving two left handed field \( \psi \) and \( \chi \) and a Higgs field \( \phi \), is given by

\[ L_{Yu} = y \bar{\psi}_L \chi_L \phi - y^* \bar{\psi}_L^T \sigma_2 \chi_L^* \phi^* \]  
(2.37)

Under a CP transformation, it becomes

\[ L_{Yu} \rightarrow y \bar{\psi}_L \chi_L \phi^{CP} - y^* \bar{\psi}_L^T \sigma_2 \chi_L^* \phi^{*CP} \]
\[ = -y \bar{\psi}_L \chi_L \phi^{*CP} + y^* \bar{\psi}_L^T \sigma_2 \chi_L^* \phi^{CP}, \]  
(2.38)

where \( \phi^{CP} \) is the CP transform of \( \phi \). It is invariant if

\[ y \phi^{CP} = y^* \phi^*. \]  
(2.39)

Since the CP transform of \( \phi \) is the complex conjugate, we conclude that CP invariance is maintained if the Yukawa couplings are real, \( y = y^* \). It follows that the standard model explicitly breaks CP because the three-family CKM matrix has just enough room to accommodate one complex phase. It also means that CP-violating effects will only appear when all three families are incorporated, since with two families, all CP-violating phases can be absorbed away. Thus we expect two possible manifestations of CP-violation. One where particles made up of the first two families are involved, in which case the CP-violation has to enter through loop diagrams involving the third family, the other at tree level with particles of all three families involved. Clearly, the study of CP-violation is a powerful probe into the structure of the standard model, where all CP-violating phenomena derive from only one parameter.
2.3.1 PROBLEMS

A. 1-) Show that with one complex Higgs doublet, one can form only one quartic invariant.

   2-) How many independent quartic invariants can be formed out of two doublets with the same hypercharge?

B. Consider all possible fermion bilinears which transform as a Lorentz vector. Sort them in terms of their baryon and lepton numbers. For each class, find the possible gauged quantum numbers of these combinations.

C. Repeat problem B for the fermion bilinears that are Lorentz invariants.

D. Suppose there were four chiral families of fermions. Find out the number of parameters needed to describe the Yukawa sector. How many $CP$-violating phases are there?

E. Using the triangle graphs, show that while neither quark nor lepton numbers are conserved, a certain linear combination survives.